Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number

Candidate Number

Pearson Edexcel International Advanced Level

Thursday 6 June 2024

Morning (Time: 1 hour 30 minutes)

Paper reference **WMA14/01**

Mathematics

International Advanced Level
Pure Mathematics P4

You must have:
Mathematical Formulae and Statistical Tables (Yellow), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over

Pearson

1. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

Find

$$\int_0^{\frac{\pi}{6}} x\cos 3x\,dx$$

giving your answer in simplest form.

(5)

$$\int_0^{\frac{\pi}{6}} x\cos 3x\,dx$$ multiplication ∴ use By Parts

$v = x \rightarrow \frac{dv}{dx} \rightarrow v' = 1$

★ Integration by Parts:
$\int vu'\,dx = vu - \int uv'\,dx$

$u = \frac{1}{3}\sin 3x \leftarrow \int dx \leftarrow u' = \cos 3x$

reverse chain rule: divide by the derivative of the bracket

Substitute into the formula for By Parts:

$$\int x\cos 3x\,dx = x\cdot\frac{1}{3}\sin 3x - \int \frac{1}{3}\sin 3x \cdot 1\,dx$$ M1

$$= \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,dx$$

$$= \frac{1}{3}x\sin 3x - \frac{1}{3}\cdot -\frac{1}{3}\cos 3x$$ $\int \sin kx\,dx = -\frac{1}{k}\cos kx + c$

$$\int_0^{\frac{\pi}{6}} x\cos 3x\,dx = \left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_0^{\frac{\pi}{6}}$$ dM1A1

$$= \left(\frac{1}{3}\left(\frac{\pi}{6}\right)\sin\left(3\cdot\frac{\pi}{6}\right) + \frac{1}{9}\cos\left(3\cdot\frac{\pi}{6}\right)\right) - \left(\frac{1}{3}(0)\cancel{\sin(3(0))}^{0} + \frac{1}{9}\cancel{\cos(0)}^{1}\right)$$ M1

$$= \frac{\pi}{18}\sin\frac{\pi}{2} + \frac{1}{9}\cancel{\cos\frac{\pi}{2}}^{0} - \frac{1}{9}$$

$$\int_0^{\frac{\pi}{6}} x\cos 3x\,dx = \frac{\pi}{18} - \frac{1}{9}$$ A1

Question 1 continued

(Total for Question 1 is 5 marks)

2. With respect to a fixed origin, O, the point A has position vector

$$\overrightarrow{OA} = \begin{pmatrix} 7 \\ 2 \\ -5 \end{pmatrix}$$

Given that

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$$

(a) find the coordinates of the point B.

(2)

The point C has position vector

$$\overrightarrow{OC} = \begin{pmatrix} a \\ 5 \\ -1 \end{pmatrix}$$

where a is a constant.

Given that $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{BC}$

(b) find the possible values of a.

(4)

(a) We know that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. We are given both $\overrightarrow{OA}$ and $\overrightarrow{AB}$, ∴ Substitute to get $\overrightarrow{OB}$.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} = \overrightarrow{OB} - \begin{pmatrix} 7 \\ 2 \\ -5 \end{pmatrix}$$

$$\overrightarrow{OB} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ 2 \\ -5 \end{pmatrix} \quad M1$$

$$\overrightarrow{OB} = \begin{pmatrix} 5 \\ 6 \\ -2 \end{pmatrix} \rightarrow (5, 6, -2) \quad \text{coordinates of } B$$

A1

P 7 4 3 1 5 R A 0 4 2 8

Question 2 continued

(a) Use the given $\vec{OC}$ and the $\vec{OB}$ we got in (a) to get $\vec{BC}$:

$$\vec{BC} = \vec{OC} - \vec{OB}$$

$$= \begin{pmatrix} a \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \\ -2 \end{pmatrix}$$

$$\vec{BC} = \begin{pmatrix} a-5 \\ -1 \\ 1 \end{pmatrix}$$

We are given that $\vec{OC}$ and $\vec{BC}$ are perpendicular, $\therefore$ dot product $=0$.
Use this fact to get the possible values of a.

Formula for dot product:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf$$

$$\vec{BC} \cdot \vec{OC} = \begin{pmatrix} a-5 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ 5 \\ -1 \end{pmatrix} = 0 \quad \text{M1}$$

$$0 = a(a-5) + 5(-1) + (-1)(1)$$
$$= a^2 - 5a - 5 - 1$$
$$0 = a^2 - 5a - 6 \quad \text{Quadratic Equation}$$

Solve the Quadratic: dM1

$$a^2 - 5a - 6 = (a-6)(a+1) = 0 \quad \text{ddM1}$$

$$(a-6)(a+1) = 0$$

$$a = 6 \qquad a = -1$$

$\therefore$ Possible values of $a = -1, 6$ A1

(Total for Question 2 is 6 marks)

3. The curve C is defined by the equation

$$8x^3 - 3y^2 + 2xy = 9$$

Implicit!

Find an equation of the normal to C at the point $(2, 5)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(7)

We need to differentiate to get the gradient of the tangent to the curve at point $(2, 5)$

multiplication → product rule

$$8x^3 - 3y^2 + 2xy = 9$$

★ Implicit differentiation:
When differentiating y, multiply by $\frac{dy}{dx}$

★ Product Rule:
$\frac{d}{dx}(uv) = uv' + u'v$

M1 dM1 $\quad 3 \cdot 8x^2 - 3 \cdot 2y \cdot \frac{dy}{dx} + 2y + 2x \cdot \frac{dy}{dx} = 0$

$\frac{d}{dx}(2xy)$

$u = 2x \quad u' = 2$

$v = y \quad v' = \frac{dy}{dx}$

$\therefore = 2y + 2x\frac{dy}{dx}$

A1 $\quad 24x^2 - 6y\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} = 0$

Substitute the Point $(2, 5)$ into the differentiated equation.

$$24(2)^2 - 6(5)\frac{dy}{dx} + 2(5) + 2(2)\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$:

$$24 \cdot 4 - 30\frac{dy}{dx} + 10 + 4\frac{dy}{dx} = 0$$

$$106 = 26\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{53}{13}$$ gradient of tangent

M1A1

The Normal is perpendicular to the tangent, $\therefore$ $m(\text{normal}) = -\frac{1}{m(\text{tangent})}$

Hence: $m(\text{normal}) = -\frac{1}{\frac{53}{13}} = -\frac{13}{53}$

Use $y - y_1 = m(x - x_1)$ to get the equation of the line

$m = \frac{13}{53}$, $x_1 = 2$, $y_1 = 5$ (given)

found above

$$y - 5 = -\frac{13}{53}(x - 2) \quad \text{M1}$$

$$53y - 265 = -13x + 26$$

$$\therefore 13x + 53y - 291 = 0$$ Equation of Normal

A1

Question 3 continued

(Total for Question 3 is 7 marks)

4.

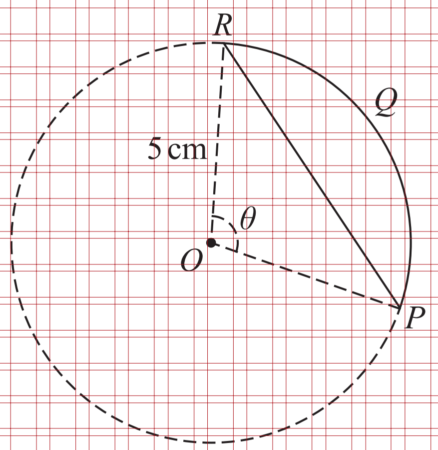

Figure 1

Figure 1 shows a sketch of a segment $PQRP$ of a circle with centre O and radius 5 cm.

Given that

- angle POR is θ radians
- θ is increasing, from 0 to π, at a constant rate of 0.1 radians per second
- the area of the segment $PQRP$ is $A\ \text{cm}^2$

(a) show that

$$\frac{dA}{d\theta} = K(1 - \cos\theta)$$

where K is a constant to be found.

(2)

(b) Find, in $\text{cm}^2\,\text{s}^{-1}$, the rate of increase of the area of the segment when $\theta = \frac{\pi}{3}$

(4)

(a) Formula for Area of a Segment:

$A = \frac{1}{2}r^2(\theta - \sin\theta)$ where: r is radius

θ is angle in rad

Hence here: $A = \frac{1}{2}\cdot(5)^2\cdot(\theta - \sin\theta)$

$A = \frac{25}{2}(\theta - \sin\theta)$

Differentiate to get $\frac{dA}{d\theta}$:

$\frac{dA}{d\theta} = \frac{25}{2}(\theta - \cos\theta)$ M1A1 $\frac{d}{dx}(\sin x) = \cos x$

hence shown, $K = \frac{25}{2}$

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Question 4 continued

(b) We are given that $\frac{d\theta}{dt} = 0.1$. "θ increasing constant rate of 0.1 rad/s"

B1

"Rate of increase of the Area of the segment" $\longrightarrow \frac{dA}{dt}$

We need to get $\frac{dA}{dt}$ from $\frac{dA}{d\theta}$ and $\frac{d\theta}{dt}$, which we found above/were given.

the $d\theta$'s cancel out

$$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$$

B1

given in question

$$= \frac{25}{2}\left[1 - \cos\frac{\pi}{3}\right] \times 0.1$$

M1

$$= \left(\frac{25}{2} - \frac{25}{4}\right) \cdot \frac{1}{10}$$

$$= \frac{25}{40} = \frac{5}{8}$$

$$\frac{dA}{dt} = \frac{5}{8}\,\text{cm}^2\text{s}^{-1}$$

A1

(Total for Question 4 is 6 marks)

P 7 4 3 1 5 R A 0 9 2 8

5.

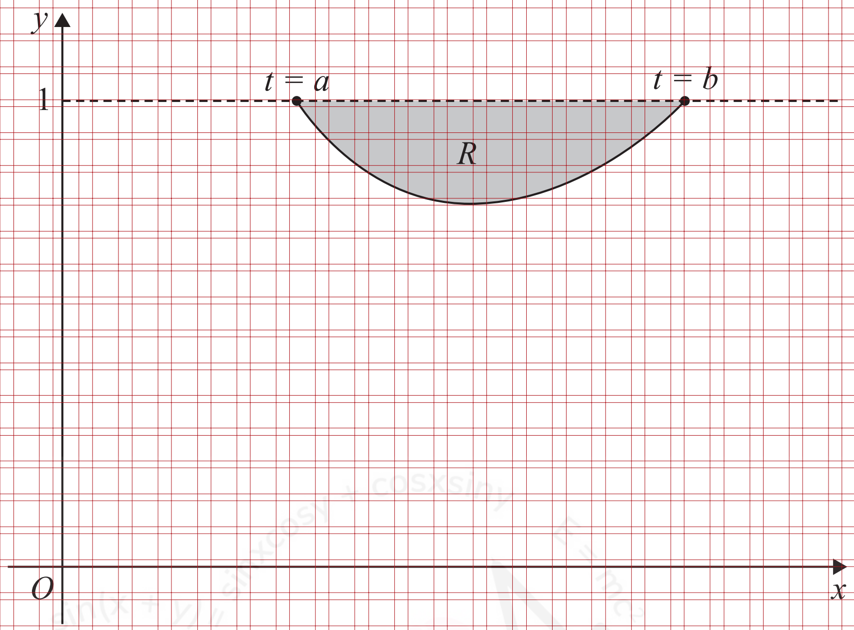

Figure 2

Figure 2 shows a sketch of the curve defined by the parametric equations

$$x = t^2 + 2t \qquad y = \frac{2}{t(3-t)} \qquad a \leqslant t \leqslant b$$

where a and b are constants.

The ends of the curve lie on the line with equation $y = 1$

(a) Find the value of a and the value of b

(2)

The region R, shown shaded in Figure 2, is bounded by the curve and the line with equation $y = 1$

(b) Show that the area of region R is given by

$$M - k\int_a^b \frac{t+1}{t(3-t)}\,\mathrm{d}t$$

where M and k are constants to be found.

(5)

(c) (i) Write $\dfrac{t+1}{t(3-t)}$ in partial fractions.

(ii) Use algebraic integration to find the exact area of R, giving your answer in simplest form.

(6)

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Question 5 continued

(a) Points a and b occur when $y=1$, $\therefore$ use the parametric equation for y to get t.

$$y=\frac{2}{t(3-t)} \Rightarrow 1=\frac{2}{t(3-t)}$$

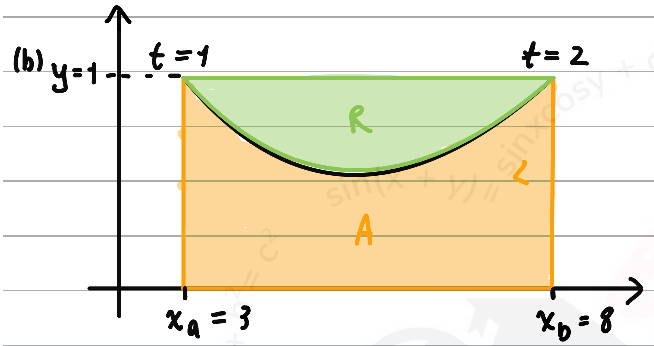
$$t(3-t)=2$$

$$3t-t^2=2 \quad \text{M1}$$

$$0=t^2-3t+2 \Rightarrow 0=(t-2)(t-1)$$

$$\therefore t=1,2 \Rightarrow a=1,\ b=2 \quad \text{A1}$$

(b)



To get the area R we need to subtract area A from the square formed by t_1, t_2, x_a and x_b.

x_a when $t=1$: $x_a=(1)^2+2(1)=3$

x_b when $t=2$: $x_b=(2)^2+2(2)=8$ B1

Area of the whole square: $(8-3)\times 1=5$ units^2.

Get A using Parametric Integration:

We already got $a=1, b=2$.

Differentiate to get $\frac{dx}{dt}$:

$$x=t^2+2t$$

$$\frac{dx}{dt}=2t+2$$

Substitute into the Formula for parametric:

★ Parametric Integration

$$\text{area}=\int_a^b y\frac{dx}{dt}\,dt$$

$$\text{area}=\int_1^2\left(\frac{2}{t(3-t)}\right)\cdot(2t+2)\,dt \quad \text{M1A1}$$

$$A=\int_1^2 \frac{4(t+1)}{t(3-t)}\,dt$$

Hence:

$$\text{area } R=5-A$$

$$\text{area } R=5-4\int_1^2\frac{t+1}{t(3-t)}\,dt \quad \text{M1 A1}$$

Question 5 continued

(c) i. Partial Fractions:

$$\frac{t+1}{t(3-t)} = \frac{A}{t} + \frac{B}{3-t} \Rightarrow t+1 = A(3-t) + B(t) \quad \text{M1}$$

Substitute in values to get A and B:

$t=0 \Rightarrow 1 = A(3) + \cancel{B(0)} \Rightarrow A = \frac{1}{3}$

$t=3 \Rightarrow 1+3 = \cancel{A(3-3)} + 3B \Rightarrow B = \frac{4}{3}$ M1

$$\therefore \frac{t+1}{t(3-t)} = \frac{1}{3t} + \frac{4}{3(3-t)} \quad \text{A1}$$

ii.

$$\text{area } R = 5 - 4\int_1^2 \frac{t+1}{t(3-t)}\,dt$$

$$= 5 - 4\int_1^2 \frac{1}{3t} + \frac{4}{3(3-t)}\,dt$$

$$= 5 - \frac{4}{3}\int_1^2 \frac{1}{t} + \frac{4}{3-t}\,dt \qquad \int \frac{1}{x}\,dx = \ln x + c$$

$$= 5 - \frac{4}{3}\Big[\ln|t| - 4\ln|3-t|\Big]_1^2 \quad \text{M1}$$

$$= 5 - \frac{4}{3}\Big[(\ln 2 - 4\ln(3-2)) - (\ln 1 - 4\ln(3-1))\Big] \quad \text{dM1}$$

$$= 5 - \frac{4}{3}\left[\ln 2 - 4\cancel{\ln 1} - \cancel{\ln 1} + 4\ln 2\right]$$

$$= 5 - \frac{4}{3}\cdot 5\ln 2$$

$$= 5 - \frac{20}{3}\ln 2 \quad \text{area } R \quad \text{A1}$$

Question 5 continued

(Total for Question 5 is 13 marks)

6. With respect to a fixed origin O, the line l_1 is given by the equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \lambda(8\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$

where λ is a scalar parameter.

The point A lies on l_1

Given that $|\overrightarrow{OA}| = 5\sqrt{10}$

(a) show that at A the parameter λ satisfies

$$81\lambda^2 + 52\lambda - 220 = 0$$

(3)

Hence

(b) (i) show that one possible position vector for A is $-15\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$

(ii) find the other possible position vector for A.

(3)

The line l_2 is parallel to l_1 and passes through O.

Given that

- $\overrightarrow{OA} = -15\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$
- point B lies on l_2 where $|\overrightarrow{OB}| = 4\sqrt{10}$

(c) find the area of triangle OAB, giving your answer to one decimal place.

(4)

(a) We are given $|\overrightarrow{OA}| = 5\sqrt{10}$.

Formula for Magnitude of a vector:

let $\overrightarrow{AB} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

$|\overrightarrow{AB}| = \sqrt{a^2 + b^2 + c^2}$

A is the general point on l_1:

$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 8 \\ -1 \\ 4 \end{pmatrix}$

$\therefore \overrightarrow{OA} = \begin{pmatrix} 1+8\lambda \\ 2-\lambda \\ 5+4\lambda \end{pmatrix}$

Get magnitude using formula:

$|\overrightarrow{OA}| = \sqrt{(1+8\lambda)^2 + (2-\lambda)^2 + (5+4\lambda)^2} = 5\sqrt{10}$ M1

M1 $1 + 16\lambda + 64\lambda^2 + 4 - 4\lambda + \lambda^2 + 25 + 40\lambda + 16\lambda^2 = 250$ Square both sides + Expand

$81\lambda^2 + 52\lambda - 220 = 0$ A1 Collect like terms

hence shown

Question 6 continued

(b) i. Solve the quadratic we found in (a)

$$81\lambda^2 + 52\lambda - 220 = 0$$

$$(81\lambda - 110)(\lambda + 2) = 0$$ factorize

$$\lambda = \frac{110}{81} \qquad \lambda = -2$$

Substitute $\lambda = \frac{110}{81}$ into $\vec{OA} = \begin{pmatrix} 1+8\lambda \\ 2-\lambda \\ 5+4\lambda \end{pmatrix}$:

$$\vec{OA} = \begin{pmatrix} 1+8\left(\frac{110}{81}\right) \\ 2-\frac{110}{81} \\ 5+4\left(\frac{110}{81}\right) \end{pmatrix} \quad \text{M1}$$

$$\vec{OA} = \begin{pmatrix} \frac{961}{81} \\ \frac{52}{81} \\ \frac{845}{81} \end{pmatrix} \quad \text{A1}$$ one possibility for $\vec{OA}$

ii. Substitute $\lambda = -2$ into $\vec{OA} = \begin{pmatrix} 1+8\lambda \\ 2-\lambda \\ 5+4\lambda \end{pmatrix}$:

$$\vec{OA} = \begin{pmatrix} 1+8(-2) \\ 2-(-2) \\ 5+4(-2) \end{pmatrix}$$

$$\vec{OA} = \begin{pmatrix} -15 \\ 4 \\ -3 \end{pmatrix} \quad \text{A1}$$ second possibility for $\vec{OA}$.

(c) We will use the formula:

$A = \frac{1}{2}ab\sin C$

to get the area, where a and b are sides and C is the size of the angle between them.

Let's first find the angle between $\vec{OA}$ and $\vec{OB}$.

Use formula:

$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$ Formula for dot product: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$

Since l_2 is parallel to l_1 and passes through $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\vec{OB} = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix}$. (this is the direction vector of l_1, since they're parallel it's the same for l_2)

Substitute:

$$\cos\theta = \frac{\begin{bmatrix} -15 \\ 4 \\ -3 \end{bmatrix}\begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix}}{\sqrt{15^2+4^2+3^2}\ \sqrt{8^2+1^2+4^2}} \quad \text{M1}$$

$$= \frac{(-15)(8)+(4)(-1)+(4)(-3)}{\sqrt{15^2+4^2+3^2}\ \sqrt{8^2+1^2+4^2}}$$

$$\theta = \cos^{-1}\left(\frac{-136}{45\sqrt{10}}\right) \Rightarrow \theta = 17.1° \quad \text{A1}$$

Question 6 continued

(c) cont.

$|\overrightarrow{OA}| = 5\sqrt{10}$

$|\overrightarrow{OB}| = 4\sqrt{10}$ into $\frac{1}{2}ab\sin C = \text{area}$:

$\theta = \cos^{-1}\left(\frac{-136}{45\sqrt{10}}\right)$

$\frac{1}{2}(5\sqrt{10})(4\sqrt{10})\sin\left(\cos^{-1}\left(\frac{-136}{45\sqrt{10}}\right)\right)$ — Plug this into your calculator M1

$= 29.4 \text{ units}^2$ A1

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Question 6 continued

(Total for Question 6 is 10 marks)

7. The current, x amps, at time t seconds after a switch is closed in a particular electric circuit is modelled by the equation

$$\frac{dx}{dt} = k - 3x$$

where k is a constant.

Initially there is zero current in the circuit. $t=0,\ x=0$

(a) Solve the differential equation to find an equation, in terms of k, for the current in the circuit at time t seconds.
Give your answer in the form $x = f(t)$.
(6)

Given that in the long term the current in the circuit approaches 7 amps,

(b) find the value of k.
(2)

(c) Hence find the time in seconds it takes for the current to reach 5 amps, giving your answer to 2 significant figures.
(3)

(a) $$\frac{dx}{dt} = k - 3x$$

$$\frac{1}{k-3x}\,dx = dt$$ Separate Variables by grouping the like terms on one side

$$\int \frac{1}{k-3x}\,dx = \int 1\,dt$$ M1

$$-\frac{1}{3}\ln|k-3x| = t + c$$ $\int \frac{1}{x}\,dx = \ln x + c$

Reverse chain rule: divide by the derivative of the bracket

M1A1

We are given that at $t=0$ (initially), $x=0$.

$$-\frac{1}{3}\ln(k-3(0)) = 0 + c$$

$$-\frac{1}{3}\ln k = c$$ value for c

M1

Substitute c back in and make x the subject:

$$-\frac{1}{3}\ln(k-3x) = t - \frac{1}{3}\ln k$$ M1

$$-\ln(k-3x) = 3t - \ln k$$

$$-3t = \ln(k-3x) - \ln k$$

$$-3t = \ln\frac{k-3x}{k}$$ apply log law: $\ln a - \ln b = \ln\frac{a}{b}$

$$e^{-3t} = \cancel{e^{\ln}}\frac{k-3x}{k} = \frac{k-3x}{k}$$

$$ke^{-3t} = k - 3x$$

$$3x = k - ke^{-3t}$$

$$x = \frac{k(1-e^{-3t})}{3}$$ A1

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Question 7 continued

(b) Given that as $t \to \infty$, $x \to 7$.

$x = \frac{k}{3}(1 - e^{-3t})$

$e^{-3t} = \frac{1}{e^{3t}}$

as $t \to \infty$, $\frac{1}{e^{3t}} \to 0$.

M1 $\therefore 7 = \frac{k}{3}(1 - 0)$ (when $t \to \infty$)

$k = 21$ A1

(c) We got that: $x = 7(1 - e^{-3t})$.

Substitute $x = 5$ and solve:

$5 = 7(1 - e^{-3t})$ M1

$\frac{5}{7} = 1 - e^{-3t}$

$e^{-3t} = 1 - \frac{5}{7}$

$e^{-3t} = \frac{2}{7}$

$\ln e^{-3t} = \ln \frac{2}{7}$ apply log law: $\ln e^x = x$

$-3t = \ln \frac{2}{7}$

$t = -\frac{1}{3} \ln \frac{2}{7}$ dM1 plug this into your calculator to get a decimal

$\therefore t = 0.42$ seconds A1

Question 7 continued

Question 7 continued

(Total for Question 7 is 11 marks)

8. $\quad f(x) = (8 - 3x)^{\frac{4}{3}} \qquad 0 < x < \frac{8}{3}$

(a) Show that the binomial expansion of f(x) in ascending powers of x up to and including the term in x^3 is

$$A - 8x + \frac{x^2}{2} + Bx^3 + \ldots$$

where A and B are constants to be found.

(4)

(b) Use proof by contradiction to prove that the curve with equation

$$y = 8 + 8x - \frac{15}{2}x^2$$

does **not** intersect the curve with equation

$$y = A - 8x + \frac{x^2}{2} + Bx^3 \qquad 0 < x < \frac{8}{3}$$

where A and B are the constants found in part (a).

(Solutions relying on calculator technology are not acceptable.)

(4)

(a) Get the formula for the Binomial Expansion from the formula booklet:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \text{, Range of validity } |x|<1$$

We see that the given expansion is in form $(a+bx)^n$, we need to manipulate to get $(1+ax)^n$.

$$(8-3x)^{\frac{4}{3}} = (8)^{\frac{4}{3}}\left(1-\frac{3}{8}x\right)^{\frac{4}{3}} \quad \text{factor out the 8.}$$

$$= 16\left(1-\frac{3}{8}x\right)^{\frac{4}{3}} \quad \text{correct form}$$

M1

Substitute into the Formula:

$$16\left(1-\frac{3}{8}x\right)^{\frac{4}{3}} = 16\left[1+\left(\frac{4}{3}\right)\left(-\frac{3}{8}x\right) + \frac{\left(\frac{4}{3}\right)\left(\frac{4}{3}-1\right)}{2!}\left(-\frac{3}{8}x\right)^2 + \frac{\left(\frac{4}{3}\right)\left(\frac{4}{3}-1\right)\left(\frac{4}{3}-2\right)}{3!}\left(-\frac{3}{8}x\right)^3 + \ldots\right] \text{ M1}$$

$$= 16\left[1 - \frac{4}{8}x + \frac{\frac{4}{9}}{2}\cdot\left(\frac{9}{64}x^2\right) + \frac{\frac{4}{3}\cdot\frac{1}{3}\cdot -\frac{2}{3}}{6}\left(-\frac{27}{512}x^3\right) + \ldots\right]$$

$$= 16\left(1 - \frac{1}{2}x + \frac{\cancel{2}^{1}}{\cancel{9}_{1}}\cdot\frac{\cancel{9}^{1}}{\cancel{64}_{32}}x^2 + \left(-\frac{\cancel{8}^{1}}{\cancel{27}_{1}\cdot 6}\right)\left(-\frac{\cancel{27}^{1}}{\cancel{512}_{64}}x^3\right) + \ldots\right)$$

$$= 16 - 8x + \frac{1}{2}x^2 + \frac{16}{6\cdot 64}x^3 + \ldots$$

$$= 16 - 8x + \frac{1}{2}x^2 + \frac{x^3}{24} + \ldots \quad \text{hence shown} \quad \text{A1A1}$$

P 7 4 3 1 5 R A 0 2 2 2 8

Question 8 continued

(a) Make an Assumption:

"Assume the curves meet, $\therefore 8x - \frac{15x^2}{2} + 8 = 16 - 8x + \frac{x^2}{2} + \frac{x^3}{24}$"

We want our assumption to be the opposite of what we are trying to prove.

M1 In the Proof, we are trying to contradict the assumption

Solve the equation of the assumption to show that there are real solutions for x.

$$8x - \frac{15x^2}{2} + 8 = 16 - 8x + \frac{x^2}{2} + \frac{x^3}{24}$$

$$8x + 8 - 16 + 8x = \frac{x^2}{2} + \frac{15x^2}{2} + \frac{x^3}{24}$$

$$16x - 8 = \frac{16x^2}{2} + \frac{x^3}{24}$$

$$0 = 8x^2 - 16x + 8 + \frac{x^3}{24}$$

complete square for quadratic

$$0 = 8(x^2 - 2x) + 8 + \frac{x^3}{24}$$

$$0 = 8(x-1)^2 - 8 + 8 + \frac{x^3}{24}$$ M1

$$0 = 8(x-1)^2 + \frac{x^3}{24}$$ A1

In this format, we can deduce that $8(x-1)^2 + \frac{x^3}{24} > 0$ for the given range $0 < x < \frac{8}{3}$

$$8(x-1)^2 \geq 0$$

and $\frac{x^3}{24} > 0$ A1

This means that $8(x-1)^2 + \frac{x^3}{24}$ would never cross the x-axis (since all values are >0) and hence, there are no real solutions for x. This is a contradiction to our assumption, hence it's proven by contradiction that the curves don't meet.

Question 8 continued

Question 8 continued

(Total for Question 8 is 8 marks)

9.

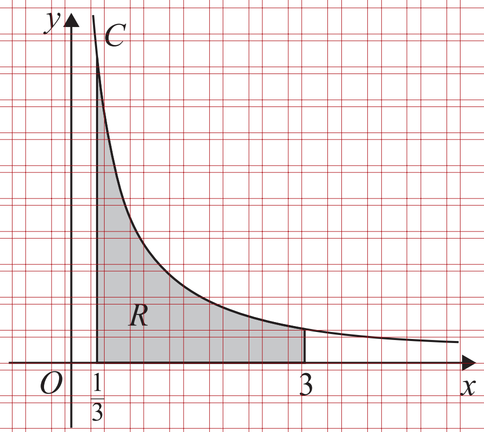

Figure 3

The curve C, shown in Figure 3, has equation

$$y = \frac{x^{-\frac{1}{4}}}{\sqrt{1+x}\left(\arctan\sqrt{x}\right)}$$

The region R, shown shaded in Figure 3, is bounded by C, the line with equation $x = 3$, the x-axis and the line with equation $x = \frac{1}{3}$

The region R is rotated through 360° about the x-axis to form a solid.

Using the substitution $\tan u = \sqrt{x}$

(a) show that the volume V of the solid formed is given by

$$k\int_a^b \frac{1}{u^2}\,\mathrm{d}u$$

where k, a and b are constants to be found.

(6)

(b) Hence, using algebraic integration, find the value of V in simplest form.

(3)

Question 9 continued

(a) Formula for Volume of Revolution around the x-axis:

$$V = \pi \int_a^b y^2 \, dx$$

where a and b are the points on the x-axis bounding the area

In our case, $a = \frac{1}{3}$ and $b = 3$, $\therefore V = \pi \int_{\frac{1}{3}}^{3} y^2 \, dx$.

$$V = \pi \int_{\frac{1}{3}}^{3} \left(\frac{x^{-\frac{1}{4}}}{\sqrt{1+x}\,(\arctan\sqrt{x})} \right)^2 dx \quad \text{— apply the power}$$

$$= \pi \int_{\frac{1}{3}}^{3} \left(\frac{x^{-\frac{1}{2}}}{(1+x)(\arctan\sqrt{x})^2} \right) dx \quad \text{B1}$$

We are given the substitution $\tan u = \sqrt{x}$. We need to get $dx = \ldots$, $x = \ldots$ and $u = \ldots$

$dx = \ldots$ $\quad \tan u = x^{\frac{1}{2}} \rightarrow \sec^2 u \frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$ Implicit Differentiation

M1A1

$$dx = \frac{\sec^2 u \, du}{\frac{1}{2} x^{-\frac{1}{2}}} \quad \text{Rearrange}$$

$x = \ldots$ $\quad (\tan u)^2 = (x^{\frac{1}{2}})^2$ Square Both sides

$\tan^2 u = x$

$u = \ldots$ $\quad u = \arctan\sqrt{x}$

Also find new Limits:

$x = \frac{1}{3}$ $\quad u = \arctan\sqrt{\frac{1}{3}} = \frac{\pi}{6}$ lower bound

$x = 3$ $\quad u = \arctan\sqrt{3} = \frac{\pi}{3}$ upper bound

B1

P 7 4 3 1 5 R A 0 2 7 2 8

Question 9 continued

Substitute everything in:

$$V = \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x^{-\frac{1}{2}}}{(1+\tan^2 u) \cdot u^2} \cdot \frac{\sec^2 u \, du}{\frac{1}{2}x^{-\frac{1}{2}}}$$ dM1

$$= \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\cancel{x^{-\frac{1}{2}}}}{\frac{1}{2}\cancel{x^{-\frac{1}{2}}}}}{u^2 \cdot \sec^2 u} \cdot \sec^2 u \, du$$

Trig. Identity: $1 = \sec^2 u - \tan^2 u$

$$= \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2\cancel{\sec^2 u}}{u^2 \cancel{\sec^2 u}} \, du$$

$$= \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2}{u^2} \, du$$

$$= 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{u^2} \, du$$ A1

$k = 2\pi$, $a = \frac{\pi}{6}$, $b = \frac{\pi}{3}$

Hence shown.

(b) Integrate what we got in (a):

$$V = 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} u^{-2} \, du$$

★ Simple Integration: $\int x^n \, dx = \frac{1}{n} x^{n+1} + c$ remember no $\pm c$ when you have limits!

$$= 2\pi \left[-u^{-1} \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$ M1

$$= 2\pi \left[-\frac{1}{\frac{\pi}{3}} - \left(-\frac{1}{\frac{\pi}{6}} \right) \right]$$ dM1

$$= 2\pi \left(-\frac{3}{\pi} + \frac{6}{\pi} \right)$$

$$= 2\cancel{\pi} \cdot \frac{3}{\cancel{\pi}}$$

Volume $= 6$ A1

(Total for Question 9 is 9 marks)

TOTAL FOR PAPER IS 75 MARKS